Regular Representation

$r: G \to Sym(X)$ $\rightsquigarrow$ $\rho: G \to GL(\mathbb{C}X)$

group action permutation representation

Pick V, $\dim V = |X| < \infty$

Pick B basis V.

Choose bijection $B \leftrightarrow X$, $u_x \in B$, $x \in X$

so $\sum_{x \in X}' a_x u_x \in V = \mathbb{C}X$

Define: $\rho: G \to GL(\mathbb{C}X)$ by

$$\rho_g(u_x) := u_{gx}$$

$x \in X$, $g \in G$ $\rightsquigarrow$ $gx \in X$

[e.g. $G = S_n$, $\rho: S_n \to GL_n(\mathbb{C})$ std rep

action S_n on $\{1, \dots, n\} \longleftrightarrow \{e_1, \dots, e_n\} \subseteq \mathbb{C}^n$]

Prop: G acts on X

$\rho : G \to GL(\mathbb{C}X)$ permutation rep

$$\Rightarrow \quad \chi_\rho(g) := |Fix(g)|$$

where $Fix(g) := \{ x \in X \mid gx = x \}$

Proof: $A = [\rho_g]_B$ $\qquad X = \{x_1, \dots, x_d\}$ $\quad B = \{u_{x_1}, \dots, u_{x_d}\}$

is a permutation matrix

$$a_{ii} = \begin{cases} 1 & \text{if } g \cdot x_i = x_i \\ 0 & \text{if } g \cdot x_i \neq x_i \end{cases}$$

$$\Rightarrow \quad Tr(A) = |Fix(g)|. \qquad \blacksquare$$

regular representation:

$X = G$, G acts on X by left mult.

$$\Rightarrow \quad L : G \to GL(\mathbb{C}G), \quad \dim = |G|$$

$$L_g(u_x) = u_{gx} \qquad g, x \in G$$

$$L_g(u_x) = u_{gx}$$

$$\chi_L(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}$$

Every irreducible G-rep is contained in L

Theorem: $\lambda^1, \ldots, \lambda^s$ = complete list of irreps
$d_k := \dim \lambda^k > 0$

Then $L \sim d_1 \lambda^1 \oplus \cdots \oplus d_s \lambda^s$

and $|G| = \sum_{k=1}^{s} d_k^2$

Proof: $\chi_k = \chi_{\lambda^k}$

$$\langle \chi_L, \chi_k \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_L(g) \overline{\chi_k(g)}$$

$\shortparallel$
mult of λ^k
in L

$$= \frac{1}{|G|} \chi_L(e) \overline{\chi_k(e)} = \frac{1}{|G|} |G| \overline{d_k}$$

$\shortparallel$
d_k

$$\chi_L = \sum_{k=1}^{s} d_k \chi_k$$ evaluate at e:

$$|G| = \chi_L(e) = \sum_k d_k \chi_k(e) = \sum_k d_k^2 .$$